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Fundamental Theorems of Calculus for Hausdorff Measures on the Real Line

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The fundamental theorems of calculus are extended to the treatment of Hausdorff measures on the real line. This includes the study of local properties of the graph of a function which is an indefinite integral with respect to Hausdorff measure as well as description of the change in the Hausdorff or Lebesgue measure of a set under a differentiable or nondifferentiable deformation.

1. INTRODUCTION

Our main purpose is to prove the following two extensions of the fundamental theorems of calculus to Hausdorff measure.

1.1. FIRST FUNDAMENTAL THEOREM OF CALCULUS FOR HAUSDORFF MEASURE. *Let m_λ be a nonatomic Hausdorff measure. Let $f\colon (a, b] \to [0, \infty)$ be an m_λ-integrable function. Let*

$$F(x) = F(a) + \int_{(a,x]} f(y)\, dm_\lambda(y).$$

Then $f(x) = D^\lambda F(x)$ almost everywhere $[m_\lambda]$.

1.2. SECOND FUNDAMENTAL THEOREM OF CALCULUS FOR HAUSDORFF MEASURE. *Let m_λ be a nonatomic Hausdorff measure. Let $F\colon [a, b] \to R$ be increasing and continuous. Let $X = \{x\colon D^\lambda F(x) = 0\}$ and $Y = \{x\colon D^\lambda F(x) = \infty\}$. If $mF(X) = mF(Y) = 0$ then*

$$F(x) = F(a) + \int_{(a,x]} D^\lambda F(y)\, dm_\lambda(y)$$

for x in (a, b).

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A key to these extensions is the outer-envelope derivative D^λ, which is discussed in Section 3. All the other terms carry their usual meanings. These theorems allow us to represent some counterexamples to the classical fundamental theorems, such as the Cantor function, as indefinite integrals.

Previous work along the lines of this paper was done by A. S. Besicovitch [1], who studied the local properties of Hausdorff-measurable sets, and by C. A. Rogers and S. J. Taylor [3–5], who obtained a characterization of increasing functions representable as indefinite integrals with respect to Hausdorff measure.

With the recognition of the importance of fractals in many areas of the physical sciences, pathological functions such as the Cantor function are seen to have physical relevance. Because of their potential as tools for the analysis of fractals, Hausdorff measure and the related concept of Hausdorff dimension have been increasingly studied in recent years; Farmer *et al.* [2] and Young [7] are just two examples. Our two theorems are intended to facilitate the use of Hausdorff measure in this context. Thus, this paper can be viewed as a timely completion of this aspect of Hausdorff measure theory.

In Section 2 we review some basic properties of Hausdorff measure and Hausdorff dimension. In Section 3 we prove the fundamental theorems. In Section 4 the fundamental theorems and the methods used in their proof are applied to the study of Hausdorff measure under deformations or coordinate changes.

2. Some Measure-Theoretic Background: Hausdorff Measures

Let λ: $(0, q) \to (0, \infty)$, $q > 0$. We define the *Hausdorff measure* associated with λ, denoted m_λ, by

$$m_\lambda V = \sup_{\varepsilon > 0} \inf \sum \lambda(mI_n),$$

where $\{I_n\}$ ranges over all coverings of the Borel set V by open intervals of length less than ε. If $\lambda(t) = t$, then the Hausdorff measure m_λ equals Lebesgue measure m. If $\lambda(t) \equiv 1$, then the Hausdorff measure m_λ equals counting measure. It can be verified that for any λ the Hausdorff measure m_λ possesses all the properties required of a measure on the Borel subsets of $(a, b]$; that is, it is positive, countably additive, and $m_\lambda \phi = 0$. With the exception of counting measure, any Hausdorff measure of interest on the real line can be obtained from a function λ which is increasing and continuous on the right with $\lambda(0) = 0$. A function with these properties we call an *index function* and in particular we call λ the index function for m_λ.

In this article we restrict ourselves to consideration of index functions λ.

It can be shown that if λ is an index function then m_λ has no atoms. Most of our results could be extended to the case $\lambda \equiv 1$ of counting measure.

We denote m_λ by $m_{[p]}$ when $\lambda(t) = t^p$ for some $0 \leqslant p \leqslant 1$. In this case the Hausdorff measure $m_{[p]}$ has the scaling property:

$$m_{[p]}\{rx: x \in V\} = r^p m_{[p]} V,$$

for any Borel set V and any $r > 0$.

It is not difficult to show that, for fixed V, $m_{[p]} V$ is a decreasing function of p. The *Hausdorff dimension* of a set V is defined to be the infimum of values p such that $m_{[p]} V = 0$.

One basic example is the standard Cantor set W, the set of points in $[0, 1]$ which have no digit 1 in their ternary expansions. It can be shown that the Hausdorff dimension of W is $\log 2/\log 3 = p$ and also that $m_{[p]} W = 1$. In fact the well-known Cantor function F, which has zero derivative outside the Cantor set, satisfies

$$m_{[p]}(W \cap [0, a)) = F(a).$$

We note here one other property of Hausdorff measure on the real line that we shall make use of.

2.1. THEOREM. *Let m_λ be a Hausdorff measure. Then for any Borel set $V \subset (a, b]$ we have*

$$m_\lambda V = \sup m_\lambda W,$$

where W ranges over all compact subsets of V with $m_\lambda W < \infty$.

Proof. See Rogers [3, Theorem 57].

3. THE FUNDAMENTAL THEOREMS FOR HAUSDORFF MEASURES

Analogs to the classical fundamental theorems of calculus for Hausdorff measure require an operator which bears the same relationship to the Hausdorff measure m_λ as the classical derivative D bears to Lebesgue measure m. Hence the following definition:

3.1. DEFINITION. Let λ be an index function. Let $F: [a, b] \to R$ be continuous and increasing. We define the *outer-envelope derivative* of F associated with λ, denoted by $D^\lambda F$, by

$$D^\lambda F(x) = \inf_{\varepsilon > 0} \sup_J \frac{F(z) - F(y)}{\lambda(z - y)},$$

where $J = (y, z)$ ranges over all open intervals containing x and of length less than ε. As a basic example, if F is the standard Cantor function and $\lambda(t) = t^p$, $p = \log 2/\log 3$, then it can be shown that $D^\lambda F(x)$ is 1 if x is in the Cantor set and 0 otherwise.

Note that $D^\lambda F(a)$ and $D^\lambda F(b)$ are undefined. As long as we restrict consideration to continuous functions and nonatomic measures, this is unimportant.

The operator D^λ was defined and used by Rogers and Taylor in a study of the representation of finite measures on Euclidean space by Hausdorff measures, which is not unrelated to our present purpose. See the references [3–5]. We shall make use of some of their results:

3.2. THEOREM. *Let $F: [a, b] \to R$ be continuous and increasing. Let λ be an index function. Then for each real u, the set $\{x: D^\lambda F(x) < u\}$ is a Borel set. Furthermore, for each real u and each positive ε, the set $\{x: mF(J) < u\lambda(mJ)$ for all open intervals J containing x with $mJ < \varepsilon\}$ is closed.*

Proof. See Rogers [3, Theorems 65, 66].

3.3. THEOREM. *Let $F: [a, b] \to R$ be continuous and increasing and let $\lambda: (0, q) \to (0, \infty)$, $q > 0$. Let $V_0 = \{x: D^\lambda F(x) = 0\}$, $V_+ = \{x: 0 < D^\lambda F(x) < \infty\}$, and $V_\infty = \{x: D^\lambda F(x) = \infty\}$. Then*

(i) *V_0, V_+, and V_∞ are Borel sets;*

(ii) *$m_\lambda V_\infty = 0$;*

(iii) *V_+ is $m_\lambda - \sigma$-finite;*

(iv) *$m_\lambda X = 0$ implies $mF(X \cap V_+) = 0$;*

(v) *X is $m_\lambda - \sigma$-finite implies $mF(X \cap V_0) = 0$.*

Proof. See Rogers [3, Theorem 67].

Remark. It is well known that there is a one-to-one correspondence between nonatomic finite Borel measures μ on $[a, b]$ and continuous increasing functions F on $[a, b]$ with $F(a) = 0$, given by $F(x) = \mu(a, x]$. We point out that (for continuous F) the inversion of this correspondence is simple:

$$\mu X = mF(X),$$

for $X \subset [a, b]$. In particular, we shall make use of this fact when μ and F are defined by integrals; if

$$F(x) = F(a) + \int_{(a,x]} f\, dv,$$

ν being a nonatomic Borel measure on $[a, b]$, not necessarily finite, then

$$mF(X) = \int_X f\, d\nu.$$

The results of this section represent a considerable amplification of Theorem 3.3, particularly concerning the size of the set $F(X \cap V_+)$.

The following sequence of lemmas will lead to our analogs for Hausdorff measure of the fundamental theorems of calculus.

3.4. LEMMA. *Let F: $[a, b] \to R$ be continuous and increasing. Let λ be an index function. Suppose $V \subset (a, b]$ is a Borel set such that $D^\lambda F(x) < u$ for x in V. Then*

$$mF(V) \leqslant u m_\lambda V.$$

Proof. If $m_\lambda V = \infty$, we have nothing to prove. If $m_\lambda V < \infty$, then choose $\varepsilon > 0$. Since $D^\lambda F(x) < u$ for x in V, we may write

$$V = \bigcup_{n=1}^{\infty} V_n,$$

where

$$V_n = \{x \in V\colon mF(J) < u\lambda(mJ) \text{ for all open intervals } J \text{ containing } x \text{ with } mJ < 1/n\}.$$

By Theorem 3.2, V_n is a Borel set for each n. For each n, let $\{J_j\colon j = 1, 2,...\}$ be a covering of V_n by open intervals such that $mJ_j < 1/n$ and

$$\sum_{j=1}^{\infty} \lambda(mJ_j) < m_\lambda V_n + \varepsilon.$$

Then $\{F(J_j)\colon j = 1, 2,...\}$ covers $F(V_n)$. Thus

$$\begin{aligned} mF(V_n) &\leqslant \sum_{j=1}^{\infty} mF(J_j) \\ &\leqslant \sum_{j=1}^{\infty} u\lambda(mJ_j) \\ &\leqslant u m_\lambda V_n + u\varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we have $mF(V_n) \leqslant u m_\lambda V_n$. Thus

$$mF(V) = \sup mF(V_n) \leqslant \sup u m_\lambda V_n = u m_\lambda V. \qquad \text{Q.E.D.}$$

We shall make use of the well-known *Vitali covering lemma*, which we state here.

3.5. VITALI COVERING LEMMA. *Let $V \subset R$ with $m^*V < \infty$, where m^* denotes Lebesgue outer measure. Suppose Θ is a collection of intervals with the property that for any x in V and any $\delta > 0$, there exists $I \in \Theta$ such that $x \in I$ and $0 < mI < \delta$. Then for any $\varepsilon > 0$, Θ contains a countable subcollection $\{I_n; n = 1, 2, ...\}$ such that*

(i) $\sum_{n=1}^{\infty} mI_n < m^*V + \varepsilon$ *and*

(ii) $m^*(V - \bigcup_{n=1}^{\infty} I_n) = 0$.

Proof. See Royden [6, p. 95].

3.6. LEMMA. *Let F: $[a, b] \to R$ be continuous and increasing. Let m_λ be a nonatomic Hausdorff measure. Suppose $V \subset (a, b]$ is a Borel set such that $u \leqslant D^\lambda F(x)$ for x in V. Then*

$$um_\lambda V \leqslant mF(V).$$

Proof. Choose $\varepsilon > 0$. Let $G(x) = F(x) + \varepsilon(x-a)/(b-a)$; then G is a homeomorphism from $[a, b]$ onto $[F(a), F(b) + \varepsilon]$ and for any Borel set $X \subset (a, b]$ we have $mG(X) \leqslant mF(X) + \varepsilon$. Moreover, $D^\lambda F(x) \leqslant D^\lambda G(x)$ for all x in (a, b).

By Theorem 2.1, $m_\lambda V = \sup m_\lambda W$, where W ranges over compact subsets of V with $m_\lambda W < \infty$; it thus suffices to consider such subsets W. For each x in W, since $u \leqslant D^\lambda F(x)$, there exists a sequence of open intervals $\{(y_n(x), z_n(x)): n = 1, 2, ...\}$ such that

(i) $x \in (y_n(x), z_n(x))$ for each n;

(ii) $e^{-\varepsilon} u\lambda(z_n(x) - y_n(x)) < G(z_n(x)) - G(y_n(x))$;

(iii) $z_n(x) - y_n(x) < \varepsilon$; and

(iv) $\lim_{n\to\infty} G(z_n(x)) - G(y_n(x)) = 0$.

The collection of intervals $\{(G(y_n(x)), G(z_n(x))): x \in W, n = 1, 2, ...\}$ satisfies the hypotheses of the Vitali covering lemma with regard to the set $G(W)$. It therefore contains a countable subcollection $\{(G(y_n), G(z_n)): n = 1, 2, ...\}$ such that

$$m\left[G(W) - \bigcup_{n=1}^{\infty} (G(y_n), G(z_n))\right] = 0$$

and

$$\sum_{n=1}^{\infty} m(G(y_n), G(z_n)) < mG(W) + \varepsilon.$$

Let

$$X = G(W) - \bigcup_{n=1}^{\infty} (G(y_n), G(z_n));$$

then X is compact and $mX = 0$. Let U be an open set containing X with $mU < \varepsilon$. The collection of intervals $\{(G(y_n(x)), G(z_n(x))): x \in X, (G(y_n(x)), G(z_n(x))) \subset U\}$ covers X; it therefore contains a finite subcover $\{(G(r_n), G(s_n)): n = 1, 2, ..., k\}$. We may further assume that this subcover is minimal. Some tedious but elementary calculations show that this minimality implies that a point of X can be contained at most two intervals $(G(r_n), G(s_n))$. We thus have

$$\sum_{n=1}^{k} m(G(r_n), G(s_n)) \leqslant 2mU < 2\varepsilon$$

Note

$$W \subset \bigcup_{n=1}^{\infty} (y_n, z_n) \cup \bigcup_{n=1}^{k} (r_n, s_n).$$

Moreover,

$$\begin{aligned} &e^{-\varepsilon} u \left(\sum_{n=1}^{\infty} \lambda(z_n - y_n) + \sum_{n=1}^{k} \lambda(s_n - r_n) \right) \\ &\leqslant \sum_{n=1}^{\infty} (G(z_n) - G(y_n)) + \sum_{n=1}^{k} (G(s_n) - G(r_n)) \\ &< mG(W) + \varepsilon + 2\varepsilon \\ &< mF(W) + 4\varepsilon. \end{aligned}$$

Thus $e^{-\varepsilon} u \inf \sum \lambda(mI_n) < mF(W) + 4\varepsilon$, where $\{I_n\}$ ranges over all coverings of V by open intervals of length less than ε. Since $\varepsilon > 0$ is arbitrary, we have $um_\lambda W \leqslant mF(W)$. Taking the supremum over compact subsets W of V, we have

$$um_\lambda V = u \sup m_\lambda W \leqslant u \sup mF(W) = umF(V). \qquad \text{Q.E.D.}$$

3.7. First Fundamental Theorem of Calculus for Haudorff Measure. *Let m_λ be a nonatomic Hausdorff measure. Let $f: (a, b] \to [0, \infty)$ be an m_λ-integrable function. Let*

$$F(x) = F(a) + \int_{(a,x]} f(y)\, dm_\lambda(y).$$

Then $f(x) = D^\lambda F(x)$ almost everywhere $[m_\lambda]$.

Proof. Since m_λ is nonatomic, F is continuous. Let $V = \{x: f(x) \neq D^\lambda F(x)\}$. We may write

$$V = \bigcup_{u,v \in Q} X_{uv} \cup \bigcup_{u,v \in Q} Y_{uv},$$

where $X_{uv} = \{x: D^\lambda F(x) < u < v < f(x)\}$ and $Y_{uv} = \{x: f(x) < u < v < D^\lambda F(x)\}$. If we show $m_\lambda X_{uv} = m_\lambda Y_{uv} = 0$ for each u and v, then we have $m_\lambda V = 0$.

By Lemma 3.4, we have $mF(X_{uv}) \leqslant u m_\lambda X_{uv}$. Since $v < f(x)$ for x in X_{uv}, we also have

$$v m_\lambda X_{uv} \leqslant \int_{X_{uv}} f(y)\, dm_\lambda(y) = mF(X_{uv}).$$

Thus $v m_\lambda X_{uv} \leqslant mF(X_{uv}) \leqslant u m_\lambda X_{uv}$. Since $mF(X_{uv}) < \infty$, we have

$$mF(X_{uv}) = m_\lambda X_{uv} = 0.$$

By Lemma 3.6, we have $v m_\lambda Y_{uv} \leqslant mF(Y_{uv})$. Since $f(x) < u$ for x in Y_{uv}, we also have

$$mF(Y_{uv}) = \int_{Y_{uv}} f(y)\, dm_\lambda(y) \leqslant u m_\lambda Y_{uv}.$$

Thus $v m_\lambda Y_{uv} \leqslant mF(Y_{uv}) \leqslant u m_\lambda Y_{uv}$. Since $mF(Y_{uv}) < \infty$, we have

$$mF(Y_{uv}) = m_\lambda Y_{uv} = 0. \qquad \text{Q.E.D.}$$

3.8. Theorem. *Let m_λ be a nonatomic Hausdorff measure. Let $F: [a, b] \to R$ be continuous and increasing. Let $V \subset (a, b]$ be a Borel set such that $0 < D^\lambda F(x) < \infty$ for x in V. Then*

$$mF(V) = \int_V D^\lambda F(x)\, dm_\lambda(x).$$

Proof. Let $\varepsilon > 0$. Let $V_n = \{x \in V: e^{n\varepsilon} \leqslant D^\lambda F(x) < e^{(n+1)\varepsilon}\}$ for integers n. By Lemma 3.4, we have $mF(V_n) \leqslant e^{(n+1)\varepsilon} m_\lambda V_n$. By Lemma 3.6, we have

$$e^{n\varepsilon} m_\lambda V_n \leqslant mF(V_n).$$

Since $mF(V) = \sum_{n=-\infty}^{\infty} mF(V_n)$, we have

$$\sum_{n=-\infty}^{\infty} e^{n\varepsilon} m_\lambda V_n \leqslant mF(V) \leqslant \sum_{n=-\infty}^{\infty} e^{(n+1)\varepsilon} m_\lambda V_n$$

or

$$e^{-\varepsilon} \sum_{n=-\infty}^{\infty} e^{(n+1)\varepsilon} m_\lambda V_n \leqslant mF(V) \leqslant e^{\varepsilon} \sum_{n=-\infty}^{\infty} e^{n\varepsilon} m_\lambda V_n.$$

Note

$$\sum_{n=-\infty}^{\infty} e^{n\varepsilon} m_\lambda V_n \leqslant \int_V D^\lambda F(x)\, dm_\lambda(x) \leqslant \sum_{n=-\infty}^{\infty} e^{(n+1)\varepsilon} m_\lambda V_n.$$

Thus

$$e^{-\varepsilon} \int_V D^\lambda F(x)\, dm_\lambda(x) \leqslant mF(V) \leqslant e^{\varepsilon} \int_V D^\lambda F(x)\, dm_\lambda(x).$$

Letting ε go to 0, we obtain the desired equality. Q.E.D.

Remark. The condition that $0 < D^\lambda F(x) < \infty$ is related to the "volume lemma" used by L.-S. Young in calculation of Hausdorff dimensions for invariant measures in dynamical systems. The connection which she establishes between the volume lemma and the Hausdorff dimension of a measure has some parallels with Theorems 3.8 and 3.9, although both her hypotheses and conclusions are weaker than ours.

Consider the special case of the preceding in which the Hausdorff measure m_λ is Lebesgue measure m and the operation D^λ is the classical derivative D. Such a theorem describes the change in the Lebesgue measure of a set under a differentiable deformation. Theorem 3.8 is a generalization of this in which the original set may have Lebesgue measure zero and the deformation may be nondifferentiable. In the next section we explore further along these lines.

3.9. Second Fundamental Theorem of Calculus for Hausdorff Measure. *Let m_λ be a nonatomic Hausdorff measure. Let F: $[a, b] \to R$ be increasing and continuous. Let $X = \{x\colon D^\lambda F(x) = 0\}$ and $Y = \{x\colon D^\lambda F(x) = \infty\}$. If $mF(X) = mF(Y) = 0$ then*

$$F(x) = F(a) + \int_{(a,x]} D^\lambda F(y)\, dm_\lambda(y)$$

for x in $(a, b]$.

Proof. Suppose first that $x \in (a, b)$. Then

$$\begin{aligned} F(x) - F(a) &= mF(a, x] \\ &= m(F(a), F(x)] - mF(X) - mF(Y) \\ &= m((F(a), F(x)] - F(X) - F(Y)) \\ &= mF((a, x] - X - Y). \end{aligned}$$

We note at this step that $F((a, x] - X - Y)$ may contain some points not in $(F(a), F(x)] - F(X) - F(Y)$, but the set of such points is countable.

From the previous theorem, we have

$$mF((a, x] - X - Y) = \int_{(a,x] - X - Y} D^\lambda F(y)\, dm_\lambda(y).$$

Thus

$$F(x) - F(a) = \int_{(a,x] - X - Y} D^\lambda F(y)\, dm_\lambda(y) \leqslant \int_{(a,x]} D^\lambda F(y)\, dm_\lambda(y).$$

Moreover, from Theorem 3.3, we have $m_\lambda Y = 0$; thus

$$\int_Y D^\lambda F(y)\, dm_\lambda(y) = 0.$$

Since $D^\lambda F(y) = 0$ for y in X, we also have

$$\int_X D^\lambda F(y)\, dm_\lambda(y) = 0.$$

Thus

$$\begin{aligned}\int_{(a,x]} D^\lambda F(y)\, dm_\lambda(y) &= \int_{(a,x] - X - Y} D^\lambda F(y)\, dm_\lambda(y)\\ &= mF((a, x] - X - Y)\\ &= F(x) - F(a).\end{aligned}$$

If $x = a$, the result is immediate. If $x = b$, then we have

$$\begin{aligned}F(b) - F(a) = \lim_{n \to \infty} F(b - 1/n) &= \lim_{n \to \infty} \int_{(a,b-1/n]} D^\lambda F(y)\, dm_\lambda(y)\\ &= \int_{(a,b]} D^\lambda F(y)\, dm_\lambda(y). \qquad \text{Q.E.D.}\end{aligned}$$

In this section we have made a mild restriction which, though not strictly necessary, gave us considerable streamlining of our definitions and proofs, namely that F be increasing. The removal of this restriction would require restatement of Definition 3.1 as well as the theorems of this section. This extension to the case where F is of bounded variation is easy, although time-consuming.

4. Hausdorff Measure of the Image Set

Let us recall the conclusion of Theorem 3.8:

$$mF(V) = \int_V D^\lambda F(x)\, dm_\lambda(X).$$

This theorem describes the Lebesgue measure of the image of a set V with known Hausdorff measure under a deformation F which is nondifferentiable such that $0 < D^\lambda F < \infty$ on V. In this section we develop some variations on Theorem 3.8 which describe $m_\chi F(V)$ in terms of $m_\lambda V$, where m_χ and m_λ are (possibly identical) Hausdorff measures.

4.1. LEMMA. *Let χ, λ be index functions and suppose there exists p such that*

$$\lim_{t \to 0} \frac{\chi(rt)}{\chi(t)} = r^p$$

for all $r > 0$. Let F: $[a, b] \to R$ be continuous and increasing. Let $V \subset (a, b]$ be a Borel set such that $D^\lambda F(x) < u$ for x in V. Then

$$m_\chi F(V) \leqslant u^p m_{\chi \circ \lambda} V.$$

Remark. It can be shown that if $g(r) = \lim_{t \to 0} \chi(rt)/\chi(t)$ exists and is a continuous function of r then there exists p such that $g(r) = r^p$. All of the commonly used index functions for Hausdorff measure satisfy the premises of this lemma.

Proof. If $m_{\chi \circ \lambda} V = \infty$, we have nothing to prove. If $m_{\chi \circ \lambda} V < \infty$, then choose $\varepsilon > 0$. Let $\eta > 0$ be sufficiently small that $\chi(ut) < e^\varepsilon u^p \chi(t)$ for $t < \eta$. For $t \in (0, q)$ let

$$\delta(t) = \sup_{|z-y| < t} |F(z) - F(y)|.$$

The function $\delta(t)$ is thus increasing and tends to zero as t goes to zero.

Since $D^\lambda F(x) < u$ for x in V, we may write

$$V = \bigcup_{n=1}^{\infty} V_n,$$

where $V_n = \{x \in V: mF(I) < u\lambda(mI)$ for all open intervals I containing x with $mI < 1/n\}$.

Let $\{I_j: j = 1, 2, ...\}$ be a covering of V_n by open intervals intersecting V_n of length less than $\inf\{1/n, \varepsilon, \delta^{-1}(\eta u)\}$ and

$$\sum_{j=1}^{\infty} \chi(\lambda(mI_j)) < m_{\chi \circ \lambda} V_n + \varepsilon.$$

Then $\{F(I_j)$: $j = 1, 2, ...\}$ covers $F(V_n)$. By the definition of V_n, $mF(I_j) < \lambda(mI_j)$, and I_j was chosen small enough that $mF(I_j) < \eta u$ so

$$\chi(mF(I_j)) = \chi\left(u \frac{1}{u} mF(I_j)\right) < e^\varepsilon u^p \chi\left(\frac{1}{u} mF(I_j)\right) < e^\varepsilon u^p \chi(\lambda(mI_j)).$$

We thus have

$$\begin{aligned}\inf \sum \chi(mJ_j) &\leqslant \sum_{j=1}^{\infty} \chi(mF(I_j)) \\ &\leqslant \sum_{j=1}^{\infty} e^{\varepsilon} u^p \chi(\lambda(mI_j)) \\ &\leqslant e^{\varepsilon} u^p m_{\chi \circ \lambda} V_n + e^{\varepsilon} u^p \varepsilon,\end{aligned}$$

where $\{J_j\}$ ranges over all coverings of $F(V_n)$ by open intervals of length less than ηu. Since $\varepsilon, \eta > 0$ are arbitrary, we have $m_\chi F(V_n) \leqslant u^p m_{\chi \circ \lambda} V_n$. Thus

$$m_\chi F(V) = \sup m_\chi F(V_n) \leqslant \sup u^p m_{\chi \circ \lambda} V_n = u^p m_{\chi \circ \lambda} V. \qquad \text{Q.E.D.}$$

4.2. THEOREM. *Let χ, λ be index functions and suppose there exists p such that*

$$\lim_{t \to 0} \frac{\chi(rt)}{\chi(t)} = r^p$$

for all $r > 0$. Let F: $[a, b] \to R$ be continuous and strictly increasing. Let $V \subset (a, b]$ be a Borel set such that $0 < DF(x) < \infty$ for x in V. Then

$$m_\chi F(V) = \int_V (DF(x))^p \, dm_\chi(x),$$

where

$$DF(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h}$$

is the classical derivative of F.

Proof. Choose $\varepsilon > 0$. Let $V_n = \{x \in V : e^{n\varepsilon} \leqslant DF(x) < e^{(n+1)\varepsilon}\}$ for integers n. By the previous lemma, with $\lambda(t) = t$, we have

$$m_\chi F(V_n) \leqslant e^{p(n+1)\varepsilon} m_\chi V_n.$$

Since $D(F^{-1})(x) < e^{(1-n)\varepsilon}$ for x in $F(V_n)$ we have

$$m_\chi V_n \leqslant e^{(p-pn)\varepsilon} m_\chi F(V_n).$$

Thus

$$e^{p(n-1)\varepsilon} m_\chi V_n \leqslant m_\chi F(V_n).$$

Since $m_\chi F(V) = \sum_{n=-\infty}^{\infty} m_\chi F(V_n)$, we have

$$\sum_{n=-\infty}^{\infty} e^{p(n-1)\varepsilon} m_\chi V_n \leqslant m_\chi F(V) \leqslant \sum_{n=-\infty}^{\infty} e^{p(n+1)\varepsilon} m_\chi V_n,$$

or

$$e^{-2p\varepsilon} \sum_{n=-\infty}^{\infty} e^{p(n+1)\varepsilon} m_\chi V_n \leqslant m_\chi F(V) \leqslant e^{p\varepsilon} \sum_{n=-\infty}^{\infty} e^{pn\varepsilon} m_\chi V_n.$$

Note

$$\sum_{n=-\infty}^{\infty} e^{np\varepsilon} m_\chi V_n \leqslant \int_V (DF(x))^p \, dm_\chi(x) \leqslant \sum_{n=-\infty}^{\infty} e^{p(n+1)\varepsilon} m_\chi V_n.$$

Thus

$$e^{-2p\varepsilon} \int_V (DF(x))^p \, dm_\chi(x) \leqslant m_\chi F(V) \leqslant e^{p\varepsilon} \int_V (DF(x))^p \, dm_\chi(x).$$

Letting $\varepsilon \to 0$, we obtain

$$m_\chi F(V) = \int_V (DF(x))^p \, dm_\chi(x). \qquad \text{Q.E.D.}$$

4.3. Corollary. *If*

$$\lim_{t \to 0} \frac{\chi(rt)}{\chi(t)} = 1$$

for all $r > 0$, *and* $0 < DF(x) < \infty$ *for* x *in* V, *then*

$$m_\chi F(V) = m_\chi V.$$

Thus we have the interesting fact that for an index function such as

$$\chi(t) = \frac{1}{\ln(1/t)},$$

the Hausdorff measure of a set is completely invariant under any differentiable deformation.

4.4. Theorem. *Let* λ, χ *be index functions and suppose there exists* p *such that*

$$\lim_{t \to 0} \frac{\chi(rt)}{\chi(t)} = r^p$$

for all $r>0$. *Let* $F\colon [a,b]\to R$ *be continuous and strictly increasing. Let* $V\subset (a,b]$ *be a Borel set such that* V *is* $m_{\chi\circ\lambda}-\sigma$*-finite and* $D^{\lambda}F(x)<\infty$ *for* x *in* V. *Then*

$$m_\chi F(V)\leqslant \int_V (D^\lambda F(x))^p\, dm_{\chi\circ\lambda}(x).$$

Proof. Choose $\varepsilon>0$. Let $V_n=\{x\in V\colon e^{n\varepsilon}\leqslant D^\lambda F(x)<e^{(n+1)\varepsilon}\}$ for positive and negative integers n, $V_{-\infty}=\{x\in V\colon D^\lambda F(x)=0\}$. By Lemma 4.1 we have

$$m_\chi F(V_n)\leqslant e^{p(n+1)\varepsilon} m_{\chi\circ\lambda} V_n.$$

Letting $u\to 0$ in Lemma 4.1, we have $m_\chi F(V_{-\infty})=0$. Thus

$$\begin{aligned} m_\chi F(V) &= m_\chi F(V_{-\infty})+\sum_{n=-\infty}^{\infty} m_\chi F(V_n)\\ &\leqslant \sum_{n=-\infty}^{\infty} e^{p(n+1)\varepsilon} m_{\chi\circ\lambda} V_n\\ &\leqslant e^{p\varepsilon}\sum_{n=-\infty}^{\infty} e^{pn\varepsilon} m_{\chi\circ\lambda} V_n. \end{aligned}$$

Note

$$\sum_{n=-\infty}^{\infty} e^{pn\varepsilon} m_{\chi\circ\lambda} V_n\leqslant \int_V (D^\lambda F(x))^p\, dm_{\chi\circ\lambda}(x).$$

Thus

$$m_\chi F(V)\leqslant e^{p\varepsilon}\int_V (D^\lambda F(x))^p\, dm_{\chi\circ\lambda}(x).$$

Letting $\varepsilon\to 0$, we obtain

$$m_\chi F(V)\leqslant \int_V (D^\lambda F(x))^p\, dm_{\chi\circ\lambda}(x). \qquad \text{Q.E.D.}$$

This theorem seems less satisfactory than Theorem 4.2, giving us only an upper bound. There are examples to show that equality is sometimes but not always attained, so that this would seem to be the best result possible.

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